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Stuart vortices in a stratified mixing layer: the Holmboe model

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Abstract. Asymptotic techniques are used to model the quasi-steady state vortices that have been observed in two-dimensional simulations of vortex roll-up in stratified shear layers. A time-independent nonlinear critical layer analysis is used to find a family of steady-state finite amplitude vortices in the Holmboe model of an inviscid stratified shear layer, with the vorticity inside closed streamlines based on the Stuart vortex. The vortices are compared to results of simulations and also an alternative model where the vorticity was constant inside closed streamlines.

Key words: asymptotics, stratified shear layers, Stuart vortices, vortex roll-up

1. Introduction

One of the most fundamental problems in fluid mechanics is the roll-up of shear layers, and one of the most fascinating aspects of the two-dimensional roll-up is that, as the shear layer rolls up, a quasi-equilibrium state emerges in the form of large vortices. Such vortices can be seen in both experiments and simulations, such as those in [1–7]. These vortices are intriguing for several reasons, not least because it appears that a highly ordered state emerges from disorder. Attempts to model these large vortices date back at least as far as the 1960's, when two alternative models, Stuart vortices [8] and Benney-Bergeron-Davis (BBD) vortices [9, 10], were published. Stuart found an exact nonlinear steady-state solution of the inviscid Euler equations, for which the stream function can be cast in the form

$$\psi = \log\left(\cosh y - \varepsilon \cos x\right), \tag{1}$$

where ε is a small parameter. This solution represents an infinite row of co-rotating vortices, and when $\varepsilon = \pm 1$, the vortices become point vortices, while when $\varepsilon = 0$, we recover unidirectional shear flow. In the form above, this stream function obeys the equation $\nabla^2\psi = \left(1-\varepsilon^2\right) \mathrm{e}^{-2\psi}$, which is a form of Liouville's equation. Since the vorticity is given by $\omega = -\nabla^2\psi$, the vorticity distribution for the Stuart vortex is

$$\omega = -\frac{1-\varepsilon^2}{(\cosh y - \varepsilon \cos x)^2}. \tag{2}$$

By contrast, the BBD solution, which marked the genesis of nonlinear critical-layer theory, was found using matched asymptotic expansions, by posing an outer solution away from the critical layer and matching it to an inner solution close to the critical layer. For time-independent inviscid two-dimensional flow, it can be shown that the vorticity must be a function of the stream function, so that

$$\nabla^2\psi = \mathcal{G}(\psi), \tag{3}$$

where $\mathcal{G}$ can be any function, with Liouville's equation being a well-known specific case. In (3), $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the two-dimensional Laplacian. While the vorticity for the Stuart vortex was given by (2), the BBD theory invoked the Prandtl-Batchelor theorem [11], which says that for steady flow at high Reynolds numbers, the vorticity distribution inside closed streamlines must be uniform. Because of this, BBD argued that the vorticity should be uniform inside the vortex cores, so that the function $\mathcal{G}$ was taken to be constant there. Maslowe [12–14] extended the BBD theory to stratified flows, arguing that the temperature as well as the vorticity should be constant inside the vortex cores. For stratified flows, (3) is replaced by a pair of equations linking the temperature T with the stream function and vorticity,

$$T = \mathcal{F}(\psi), \quad \nabla^2 \psi = \mathcal{G}(\psi) + \mathrm{J}_0 y \mathcal{F}^{'}(\psi), \tag{4}$$

where J_0 is the overall Richardson number, which measures the ratio of buoyancy to inertia in the flow, so that in [12–14], both $\mathcal{F}$ and $\mathcal{G}$ were taken to be constant inside the vortex cores.

One early problem with the BBD theory was that the vorticity was discontinuous across the edges of the cat's eyes, which BBD claimed would be smoothed out by a viscous layer at the edge of the cat's eyes. However, the analysis at the corners of the cat's eye led to a Wiener-Hopf problem, and Brown and Stewartson [15] later showed that this Wiener-Hopf problem did not have a solution with the correct behavior, meaning that the discontinuity could be not be removed. Haberman [16, 17] showed that the problems with the BBD solutions could be removed if certain mean flow distortions were added to the problem, and we include those same mean flow corrections in our analysis here as series about the critical layer. In Haberman's solutions, the velocity and vorticity were continuous, but gradients of vorticity were not. In a stratified fluid, the velocity and temperature must be continuous, but gradients of the temperature need not [17]. Because of the term $\mathrm{J}_0 y \mathcal{F}^{'}(\Psi)$ in the expression for $\nabla^2 \psi$ in (4), it follows for a stratified fluid that if gradients of the temperature need not be continuous, then neither does the vorticity.

A second problem with the BBD solutions was also pointed out by Stewartson [18, 19]. In order for vorticity to homogenize inside closed streamlines, as it is assumed to do in the BBD solution where the Prandtl-Batchelor theorem was invoked, some viscosity must be present. Stewartson argued that the cat's eyes would decay before the vorticity had become fully homogenized, and indeed it is known that such a situation occurs for the Blasius boundary layer [20, 21]. However, since structures resembling the BBD solutions were reported in simulations such as those of [3], it was widely assumed that the BBD solutions must be correct. Klaassen and Peltier [1] later pointed out that the cat's eyes structures observed in simulations also resembled Stuart vortices, and this can be seen upon close examination of the original paper of [3]. Although the vorticity contours in [3] resemble both the BBD solutions and Stuart vortices, the authors describe the 'topography' of the vorticity field as consisting of a high rounded hill (the core) along with high ridges (the braids), with the core and braids separated by low passes. It is this hill-like structure of the vorticity inside the core, which [3] attribute to the coalescence of the vorticity into relatively compact cores, that suggests that the flow inside the core is the Stuart vortex rather than the BBD solutions, for which the vorticity field would be a plateau rather than a round hill. In contrast to this, [3] describe the density field inside the core as becoming nearly constant after the climax state, which they attribute to diffusion smoothing away the density contrasts there. In [3, Figure 15(b)], the density on a section through the core is plotted. For the latest time shown in that figure, the density appears to be almost constant inside the core, then suddenly begins to change once the core boundary has been crossed, and in fact the density profile in and near the core resembles the one which we present later in Figure 3.

Similar results can be found in the simulations of [2], who calculated the quasi-equilibrium states in order to study their stability. In their Figure 3, they present surface plots of the vorticity field. As in [3], the vorticity field consists of a smooth round hill (the core) along with high ridges (the braids), with the two separated by a trough, which they attribute to the entrainment of irrotational fluid into the core. The height of the braid ridges increases with J_0, which they attribute to the fact that vorticity is generated in the braids by baroclinic temperature gradients. For $J_0 \geq 0{\cdot}08$ in their simulations, the braid vorticity exceeds that in the core, so that the ridges are higher than the hill. Contours of the temperature field are plotted in their Figure 2, and again strong gradients can be seen in the braids, which they label superadiabatic stratification, with very few contours inside the core, indicating that the temperature is nearly uniform inside the core.

In later simulations, Sommeria *et al.* [4] confirmed that in the unstratified case the vorticity field in the quasi-equilibrium states resembled the Stuart vortex rather than the homogenized cores of the BBD solutions, and suggested that the Stuart vortex was a preferred state on entropy grounds. In their Figures 2–4, they show the roll-up of vorticity in the unstratified case. Because their contours are labelled, it is possible to discern from their simulations that after the climax, the vorticity decays to a smooth state resembling a round hill. In their Figures 8–10, [4] Sommeria *et al.* present a scatter-plot of the vorticity against the stream function for the steady state. If the vorticity distribution were exactly that of the Stuart vortex, it would obey Liouville's equation, and a plot of the log of the vorticity against the stream function should be a straight line. The authors of [4] claim that, at high Reynolds numbers, inside the core there is excellent agreement between their scatter-plots and a straight line, so that the vorticity distribution in the equilibrium states is the Stuart vortex.

Finally, Wang and Maxey [6] gave a kinematical description of the mixing process. They noted that in the quasi-steady state, the density field is nearly homogenized while the vorticity field develops into a well-defined structure similar to the Stuart vortex, differing from the homogeneous shear layer due to the presence of strong braids. In their Figure 1, they plot contours of density and vorticity, and their use of color in this figure enables the structure of the vorticity and temperature fields to be clearly seen, with once again the vorticity developing into a round hill surrounded by the ridge-like braids. Wang and Maxey attributed this to the vorticity layer being stretched and wrapped into a spiral structure and then diffusion acting to smooth the vorticity and density in between the spirals, and they show a cartoon of this process in their Figure 4.

The simulations discussed above suggest that rather than the vorticity homogenizing inside the vortex cores, an alternative scenario, presented by Rhines and Young [22] in a different context, occurs. Rhines and Young argued that homogenization of vorticity within closed streamlines will take place in two stages: a rapid phase dominated by shear-augmented diffusion in which the vorticity will tend to a smooth but not necessarily uniform distribution, and a second much slower phase in which the process of homogenization will be completed. The simulations presented in [4], suggest that the first, rapid, phase postulated by Rhines and Young will take place during vortex roll-up in a shear layer, but that, as suggested by Stewartson, the vortices will decay before the second stage has occurred, so that the second stage would only occur if forcing were present. Thus, the vorticity inside the vortex cores will have a smooth but non-uniform distribution, and we believe, on entropy grounds, that the Stuart vortex is that distribution. For a stratified flow, the same arguments apply to the temperature, but since that is an odd function for the flow considered here (whereas the vorticity is an even one), the smooth distribution will be zero. This last point can be arrived at from either a physical argument or symmetry. The physical argument involves considering the distribution of the temperature as the vortices roll up. Initially, relative to a reference temperature, the temperature will have one sign above the critical layer and the opposite sign below. As the vortices roll-up, this remains true outside the vortex cores, but inside the cores, there will be alternating spirals of hot and cold fluid, and diffusive effects will smooth this distribution to zero during the first rather than the second of the two homogenization stages described above. Alternatively, for the flow considered here, we know that the temperature is an odd function while the stream function is even. However, we know from (4) above that the temperature is a function of the stream function, $T = \mathcal{F}(\psi)$, so if the stream function is even, the temperature must be also. Since we require that the temperature be both odd and even, it follows that it must be zero inside the core. Outside the core, we can satisfy symmetry by taking $T = \mathcal{F}_+(\psi)$ above the core and $T = \mathcal{F}_-(\psi)$ below, with $\mathcal{F}_- = -\mathcal{F}_+$.

We turn now to the problem considered here, the Holmboe model of a stratified shear layer for which the base velocity and base temperature are both $\tanh y$. This model is named for the researcher who first studied it [23, 24]. Until recently, some confusion has accompanied this model, because linear theory suggests that disturbances should have a strong phase shift across the critical layer [25] so that disturbances should resemble the cock-eyed cat's eyes of G.I. Taylor [26] rather than the Kelvin's cat's eyes that appear in unstratified flows. By contrast, simulations (*e.g.* [3, 2, 5, 6]) suggest that stratified and unstratified shear layers are not significantly different, the principal difference being that the braids are more intense in the stratified case. Recent simulations by Staquet [5] have suggested that the Miles phase shift appears during a secondary stage of development after the big vortices have developed. Because of this we [27] have argued that during the primary roll-up there is no phase shift, and the stratification should be treated as a small parameter within the context of a nonlinear expansion, as was done by Shukhman and Churilov [28] who considered the nonlinear evolution of the Holmboe model. Because of this, we shall use the slightly stratified approach here: this is justifiable on physical grounds as even Richardson numbers as large as 0·1 (typical of numerical simulations) can be regarded as small in a weakly nonlinear expansion.

There are two main motivations for the current work. Firstly, the recent work by Mallier [27] and Staquet [5] mentioned above, suggests that the slightly stratified approach is the correct one to take, and secondly, the recent paper by Barcilon and Drazin [29] which indicates that there is interest in the techniques used in this analysis, and which we have previously applied to the Garcia model of a stratified shear layer [30]. In addition, we have previously considered a shear layer on a beta-plane [31], where we took a similar approach and assumed that β was small, corresponding to a weakly rotating fluid, while Gatski [32] has embedded a row of Stuart vortices in a channel flow.

The rest of the paper is as follows: in Section 2, we present our analysis and find the steady state vortex solutions. Essentially, our analysis involves posing an outer expansion (Section 2.1) in the main body of the fluid away from the critical layer (where the base velocity is equal to the phase velocity of the disturbance) and an inner expansion (Section 2.2) near to the critical layer, using rescaled variables, and matching the two expansions together. For the outer expansion, we use the two-dimensional Euler equations (which are inviscid and incompressible) in the vorticity-stream function formulation together with the Boussinesq approximation, which neglects the variation of the density except in the buoyancy term. We suppose that we have superimposed a small, spatially periodic, disturbance on the base flow and expand in powers of the amplitude; this expansion becomes disordered at the critical layer where it is necessary to pose an inner expansion using stretched coordinates. In this inner solution, rather than assume that vorticity and temperature are uniform (homogenized) inside the closed cat's eyes (as was done in the BBD theory and its extensions by Maslowe and Haberman), we use the Stuart vortex as our guide, for the reasons outlined above. For comparison purposes, we will present contours of the BBD solution alongside the present solution in Figure 1. Finally, in Section 3, we will make some concluding remarks.

2. Analysis

2.1. Outer expansion

The starting point of our analysis is the set of equations governing the motion of an incompressible stratified fluid in two dimensions under the influence of gravity,

$$\nabla^2\psi_t - J(\psi, \nabla^2\psi) + \mathrm{J}_0 T_x = 0, \qquad T_t - J(\psi, T) = 0,$$

where we have assumed that the flow is inviscid with no thermal diffusivity, and we have used the Boussinesq approximation, meaning that we have neglected the variation of temperature, and thereby density, in the first equation, except in the buoyancy term, $\mathrm{J}_0 T_x$. In (5), $J(a,b) = \partial(a,b)/\partial(x,y)$ denotes a Jacobian, ψ is the stream function, T is the temperature and J_0 is the overall Richardson number, which measures the ratio of buoyancy to inertia. In terms of the stream function, the velocity is given by $\underline{u} = (\psi_y, -\psi_x, 0)$. We also have an equation of state of the form $\rho = 1 - \beta_* T_*(T-1)$, where T_* is a reference temperature and β_* is the coefficient of thermal expansion, with $\mathrm{J}_0 = g\beta_*$. This equation of state links the density ρ to the temperature T. We shall seek stationary solutions that are both periodic in $\zeta = \alpha x$, where α is the wavenumber, and independent of time, so that the terms $\nabla^2\psi_t$ and T_t in (5) are identically zero. To find these stationary solutions, we shall pose an outer expansion

$$\psi \sim \psi_0 + \varepsilon\,(\psi_{11}\cos\zeta + \psi_{10}) + \varepsilon^2\,(\psi_{22}\cos 2\zeta + \psi_{21}\cos\zeta + \psi_{20}) + \cdots, \tag{5}$$

with a similar expansion for T. This outer expansion is valid in the main body of the fluid away from the critical layer, which for this problem is located at $y = 0$ because the phase velocity of ψ given by (5) is equal to zero. As we approach $y = 0$, we will find that the expansion becomes disordered, and a separate inner expansion is necessary for y close to 0. In the expansion (5), the ψ_{nm} and T_{mn} are functions of y only, not of ζ or t, while the base stream function and temperature are $\psi_0 = \log\cosh y$ and $T_0 = \tanh y$ respectively. With the non-dimensionalization used here, both T_0 and the base velocity $u_0 = \psi_0'$ tend to 1 as $y \to \infty$ and -1 as $y \to -\infty$. The terms ψ_{n0} represent mean flow distortions, and cannot be determined from the outer expansion, and these are the terms which Haberman [16, 17] found necessary to correct the problems with the BBD solutions which were discussed in Section 1. Although in this study we are describing the outer expansion initially, followed by the inner expansion, in reality, the form of the mean flow distortions and the order at which they first enter into the expansion, $\mathcal{O}(\varepsilon)$, is determined by iterating between the inner and outer expansions, meaning that the solution in the inner region tells us that the mean flow terms are required in the outer expansion and that they enter at $\mathcal{O}(\varepsilon)$ in the outer expansion. These mean-flow distortions are generated by nonlinear interactions *inside* the critical layer where nonlinear effects enter at leading order, and for matching purposes, it is necessary to include them in the outer expansion also. In our analysis, we will include the mean flow distortions in the outer expansion as series about the critical layer. The coefficients of these series can be recovered from the inner rather than the outer expansion.

In addition to expanding the stream function and temperature, we also expand the wavenumber α away from neutral, writing $\alpha^2 = \alpha_0^2 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots$. By substituting the outer expansion (5) in the governing equations (5), we get a series of equations at successive powers of ε. At leading order, $\mathcal{O}(\varepsilon)$, we recover a form of the well-known Taylor-Goldstein equation, which governs the linear stability of the flow,

$$\psi_{11}'' + \left[\alpha_0^2 - 2\text{sech}^2 y - \text{J}_0 \,\text{csch}^2 y\right] \psi_{11} = 0. \tag{6}$$

Equation (6) has a solution

$$\psi_{11} = \text{sech}^{\alpha_0} y \tanh^{1-\alpha_0} y, \tag{7}$$

which satisfies the boundary condition that $\psi_{11} \to 0$ as $y \to \pm\infty$, provided that the Richardson number and wavenumber obey the relation

$$\text{J}_0 = \alpha_0(1 - \alpha_0), \tag{8}$$

with the temperature perturbation given by

$$T_{11} = 2\psi_{11}\text{csch}2y = \text{sech}^{2+\alpha_0} y \tanh^{-\alpha_0} y. \tag{9}$$

For $\text{J}_0 < 1/4$, (8) has two roots for α_0, $\alpha_0 = 1/2 \pm \sqrt{1/4 - \text{J}_0}$, and consequently two neutral modes, one for each value of α_0. For $\text{J}_0 > 1/4$, (8) has no real roots for α_0, meaning that there are no neutral inviscid eigenmodes for $\text{J}_0 > 1/4$. A theorem due to Miles [25] and Howard [33] states that a sufficient condition for the stability of a stratified shear layer is that $\text{Ri}(y) > 1/4$ everywhere in the flow, where $\text{Ri}(y) = \text{J}_0 T_0'/\psi_0''^2$ is the local (or gradient) Richardson number. For the Holmboe model, we have $\text{Ri}(y) = \text{J}_0 \cosh^2 y$, which predicts stability, and therefore the absence of neutral modes, for $\text{J}_0 > 1/4$.

In the solution above, both ψ_{11} and T_{11} are complex for $y < 0$, which is not allowed. Because of this, Miles [25] has shown that it is necessary to include a phase shift of $-(1-\alpha_0)\pi$ across the critical layer, meaning that the phase ζ is replaced by $\zeta - (1 - \alpha_0)\pi$ when $y < 0$. This phase shift can derived by reintroducing viscosity and then taking the limit as the viscous effects tend to zero. With the phase shift present, the flow pattern should consist of cock-eyed cat's eyes [34]. However, in actuality, Kelvin's cat's eyes are observed in simulations, which indicates that the phase shift in not present during roll-up. Staquet [5] has suggested that the phase shift found by Miles, although absent during the initial roll-up stage, does eventually appear during a much later stage of the flow, when a secondary instability occurs involving both the modes mentioned above. In the present work, we are modeling the large-scale vortices observed during the primary roll-up rather than the later stage observed in [5], and therefore the phase shift will not be present in our analysis. In order for the phase shift to vanish, we require that $\alpha_0 = 1$, which corresponds to $\text{J}_0 = 0$. To accommodate this in our analysis, we will suppose that the Richardson number is small, $\mathcal{O}(\varepsilon)$, so that the effects of stratification are absent at leading order. Because of this, we will restrict our analysis to a slightly stratified fluid, and set $\text{J}_0 = \varepsilon\text{J}_1$. With this restriction, the leading-order solution given in (7–9) reduces to $\alpha_0 = 1$, $\psi_{11} = \text{sech}\ y$ and $T_{11} = \text{sech}^3 y\coth\ y$, which is still singular. We also have a mean flow correction at this order,

$$\begin{aligned} \psi_{10} &\sim \psi_{10c} + \frac{\psi''_{10c} y^2}{2} + \frac{\psi''''_{10c} y^4}{24} + \cdots , \\ T_{10} &\sim H_{10} y^{-1} + \theta'_{10c} y + \frac{\theta'''_{10c} y^3}{6} + \cdots . \end{aligned} \tag{10}$$

Although it is necessary to include these terms in the outer expansion, they are actually generated by nonlinear interactions inside the critical layer, and the coefficients must be determined from the inner rather than the outer expansion.

At subsequent orders, using the operator,

$$\mathcal{L}_m \psi_{nm} \equiv \psi''_{nm} + \left(2\mathrm{sech}^2 y - 1\right) \psi_{nm}, \tag{11}$$

we find that at $\mathcal{O}\left(\varepsilon^2\right)$, the fundamental obeys

$$\mathcal{L}_1 \psi_{21} = \left(\alpha_1 - \mathrm{J}_1 \mathrm{csch}^2 y + \psi'''_{10} \mathrm{coth} y - 4\psi'_{10} \mathrm{csch} 2y\right) \psi_{11}. \tag{12}$$

This equation must be solved as a series in y (valid in the limit $|\ y\ |$ small) since we know ψ_{10} only as a series (10), with

$$\begin{aligned} \psi_{21} &\sim \mathrm{J}_1 \log |\ y\ | + B_{21} - \frac{1}{2} \mathrm{J}_1 y^2 \log |\ y\ | + \mathcal{O}\left(y^2\right), \\ T_{21} &\sim -\frac{H_{10}}{y^3} + \frac{\mathrm{J}_1 \log |\ y\ |}{y} + y^{-1} \left(B_{21} + \frac{H_{10}}{6} + T'_{10c} - \psi''_{10c}\right) + \mathcal{O}(y), \end{aligned} \tag{13}$$

with B_{21} a constant. We also have a harmonic term, with

$$\mathcal{L}_2 \psi_{22} = \mathrm{sech}^2 y \psi_{11}^2, \tag{14}$$

with solution

$$\psi_{22} = -\frac{1}{4} \mathrm{sech}^2 y, \qquad T_{22} = -\frac{1}{4} \left(3 + \mathrm{csch}^2 y\right) \mathrm{csch}\ y\ \mathrm{sech}^3 y, \tag{15}$$

and once again, it is necessary to include mean flow distortions in the form of series,

$$\begin{aligned} \psi_{20} &\sim \psi_{20c} + \frac{\psi''_{20c} y^2}{2} + \frac{\psi''''_{20c} y^4}{24} + \cdots , \\ T_{20} &\sim H_{20} y^{-3} + H_{21} y^{-1} + \theta'_{20c} y + \cdots , \end{aligned} \tag{16}$$

where the coefficients in the series may be determined from the inner rather than the outer expansion. At the next order $\mathcal{O}\left(\varepsilon^3\right)$, we have

$$\begin{aligned} \mathcal{L}_1 \psi_{31} &= \psi_{11}^3 \mathrm{sech}^2 y - 4\psi_{11}^2 \psi'_{11} \mathrm{csch} 2y + \psi_{11} \left(\alpha_2 + 4\psi'_{20} \mathrm{csch} 2y + \psi'''_{20} \coth\ y + 2\psi_{22} \mathrm{sech}^2 y\right) \\ &\quad + 2\psi_{11} \left(\mathrm{J}_1 \cosh y \psi'_{10} - \psi'_{10}\right) \mathrm{csch}^2 y - \left(\psi'_{10} \psi'''_{10} + \mathrm{J}_1 T'_{10}\right) \coth^2 y \\ &\quad + \psi_{21} \left(\alpha_1 + \psi'''_{10} \coth\ y + 4\psi'_{10} \mathrm{csch} 2y - \mathrm{J}_1 \mathrm{csch}^2 y\right), \\ \mathcal{L}_2 \psi_{32} &= 2\psi_{21} \psi_{11} \mathrm{sech}^2 y + \psi_{22} \left(4\alpha_1 - \mathrm{J}_1 \mathrm{csch}^2 y + \psi'''_{10} \coth\ y + 4\psi'_{10} \mathrm{csch} 2y\right) \\ &\quad + \frac{\psi_{11}^2}{2 \sinh^2 y} \left[\frac{1}{2} \psi''''_{10} \cosh^2 y - \frac{1}{2} \psi'''_{10} \mathrm{coth} y + \psi''_{10}\right] \\ &\quad + \frac{\psi_{11}^2}{2 \sinh^3 y} \left(4 \cosh^2 y - 3\right) \left(\frac{\mathrm{J}_1}{2 \sinh y} - \frac{\psi'_{10}}{\cosh y}\right), \\ \mathcal{L}_3 \psi_{33} &= \psi_{11} \mathrm{sech}^2 y \left[2\psi_{22} - \frac{\psi_{11}^2}{3}\right]. \end{aligned} \tag{17}$$

The last equation in (17) has a solution $\psi_{33} = \frac{1}{12}\text{sech}^3 y$ and $T_{33} = \frac{1}{8}\text{sech}^3 y\ (\coth^5 y + \coth y - \tanh y)$. The other two equations in (17) must be solved as series, because they involve functions such as ψ_{10} which we know only as series, with

$$
\begin{aligned}
\psi_{31} &\sim \frac{1}{6}\mathrm{J}_1 H_{10} y^{-2} + \frac{1}{2}\mathrm{J}_1^2 (\log \mid y \mid)^2 + \mathrm{J}_1 \left(\mathrm{J}_1 + B_{21} + \theta'_{10c} - 2\psi''_{10c}\right) \log \mid y \mid + \mathcal{O}(1)\,, \\
\psi_{32} &\sim \frac{1}{8}\mathrm{J}_1 \theta_{10c} y^{-2} - \frac{1}{24}\mathrm{J}_1 \log \mid y \mid + \mathcal{O}(1)\,, \\
T_{31} &\sim \left(-\frac{3}{8} - 3H_{20}\right) y^{-5} - \mathrm{J}_1 H_{10} y^{-3} \log \mid y \mid + \mathcal{O}\left(y^{-3}\right), \\
T_{32} &\sim \frac{3}{4} H_{10} y^{-5} - \frac{1}{2}\mathrm{J}_1 H_{10} y^{-3} \log \mid y \mid + \mathcal{O}\left(y^{-3}\right),
\end{aligned}
\tag{18}
$$

along with another mean flow correction,

$$
\begin{aligned}
\psi_{30} &\sim G_{30} y^{-2} + \psi_{30c} + \frac{\psi''_{30c} y^2}{2} + \cdots\,, \\
T_{30} &\sim H_{30} y^{-5} + H_{31} y^{-3} + H_{32} y^{-1} + \cdots\,.
\end{aligned}
\tag{19}
$$

2.2. Inner expansion

It should be noted that many of the terms in the outer expansion are singular at the critical layer, $y = 0$, including the $\mathcal{O}(\varepsilon)$ temperature terms. These singularities are caused by the expansion becoming disordered near the critical layer, and to remedy this, we shall introduce stretched variables $y = \varepsilon^{1/2} Y$, $\psi = \varepsilon \Psi$ and $T = \varepsilon^{1/2}\Theta$ in the critical layer. The scaling for Y is chosen so that nonlinear terms enter at leading order inside the critical layer, while the scalings for Ψ and Θ can be deduced by rewriting the outer solution in terms of the stretched variable Y.

Returning to our original equations, the steady-state form of (5) admits a solution of the form

$$
T = \mathcal{F}(\psi), \qquad \nabla^2 \psi = \mathcal{G}(\psi) + \varepsilon \mathrm{J}_1 y \mathcal{F}'(\psi). \tag{20}
$$

In the rescaled coordinates, (20) becomes

$$
\begin{aligned}
&\Theta = \varepsilon^{-1/2} \mathcal{F}(\varepsilon\Psi), \\
&\Psi_{YY} + \varepsilon\alpha^2 \Psi_{\zeta\zeta} = \mathcal{G}(\varepsilon\Psi) + \varepsilon^{1/2} \mathrm{J}_1 Y \frac{d\mathcal{F}}{d\Psi}(\varepsilon\Psi).
\end{aligned}
\tag{21}
$$

Because we are assuming that the density is homogenized inside but not outside the vortex cores, the functions $\mathcal{F}$ and $\mathcal{G}$ in (20, 21) will have different forms inside and outside the cores. Inside the vortex cores, we assume that the density (but not the vorticity) is homogenized, so that the temperature is given by $T = 0$ with $\mathcal{F} \equiv 0$, while the vorticity follows from $\nabla^2\psi = \mathcal{G}_s(\psi)$, where $\mathcal{G}_s$ is a function that will be based on the Stuart vortex. Since we will have different solutions inside and outside the vortex cores, it will be necessary to match them together, as well as matching the inner and outer solutions together, and Haberman [17] has shown that the stream function, velocity and temperature (but not necessarily the vorticity or the temperature gradient) must be continuous across the edge of the cores.

Using the rescaled coordinates introduced above, we may write the governing equations (5) as follows:

$$J_{CL}\left(\Psi, \Psi_{YY} + \varepsilon\alpha^2\Psi_{\zeta\zeta}\right) = \varepsilon \mathrm{J}_1 \Theta_\zeta, \qquad J_{CL}\left(\Psi, \Theta\right) = 0, \tag{22}$$

with the Jacobian $J_{CL}(a, b) = \partial\left(a, b\right)/\partial(\zeta, Y)$. Once again, we shall pose an expansion in ε,

$$\Psi \sim \Psi_0 + \varepsilon\left[\log\varepsilon\Psi_{1l} + \Psi_1\right] + \varepsilon^2\left[(\log\varepsilon)^2\,\Psi_{2ll} + \log\varepsilon\Psi_{2l} + \Psi_2\right] + \cdots, \tag{23}$$

with a similar expansion for Θ. The form of this expansion is suggested by the outer expansion rewritten in the inner variables. Using the expansion (23) in the governing equations, at leading order, we get

$$J_{CL}\left(\Psi_0, \Psi_{0YY}\right) = 0, \tag{24}$$

which has a general solution

$$\Psi_{0YY} = \mathcal{G}_0\left(\Psi_0\right) \tag{25}$$

for any function $\mathcal{G}$. This solution (25) could also have been obtained by expanding (21, the general solution of the rescaled governing equations (22), as a series in ε. The Equation (25) for Ψ_{0YY} is a very different kind of equation from those found in the outer expansion, where the analysis involved solving a series of nonhomogeneous Rayleigh equations. In order to solve (25), and similar subsequent equations, we will use the outer solution as a guide. At leading order, our outer expansion written in inner variables becomes

$$\Psi_0 = \cos\zeta + \frac{Y^2}{2} + \psi_{10c}. \tag{26}$$

Since this satisfies (25), we will use it as our inner solution. If we set $\psi_{10c} = -1$, we have

$$\Psi_0 = \cos\zeta + \frac{Y^2}{2}, \tag{27}$$

which is the same as the leading-order term of the Stuart vortex stream function (1) rewritten in the inner variables, and also the same as the BBD solution. At the next order, $\mathcal{O}\left(\varepsilon\log\varepsilon\right)$, we have

$$J_{CL}\left(\Psi_0, \Psi_{1lYY}\right) = 0, \tag{28}$$

which has a solution

$$\Psi_{1lYY} = \mathcal{G}_{1l}\left(\Psi_0\right). \tag{29}$$

Once again, this expression could also have been obtained from an expansion of (21). This term appears because of the presence of logs in the outer expansion. Our outer expansion written in inner variables gives

$$\Psi_{1l} = \frac{\mathrm{J}_1}{2}\left(\cos\zeta - 1\right). \tag{30}$$

As this satisfies (29), we use this term for our inner solution. This term did not appear in Stuart's solution, and drops out when $\mathrm{J}_1 = 0$. We now turn to the temperature. At $\mathcal{O}\left(\varepsilon^0\right)$, we have

$$J_{CL}\left(\Psi_0, \Theta_0\right) = 0, \tag{31}$$

which has a solution

$$\Theta_0 = \mathcal{F}_0\left(\Psi_0\right), \tag{32}$$

which again could have been obtained from (21). The function $\mathcal{F}_0$ is unknown and can be found by reintroducing viscosity inside the critical layer. We note at this point that Θ_0 includes all the harmonics, as happened for the Garcia model [30] and the study by Kelly and Maslowe [12]. With the reintroduction of viscosity, the outer equations (5) become

$$\begin{aligned} J(\psi^{(v)}, \nabla^2\psi^{(v)}) - \mathrm{J}_0 T_x^{(v)} &= -\mathrm{Re}^{-1}\nabla^4\psi^{(v)}, \\ J(\psi^{(v)}, T^{(v)}) &= -\mathrm{Re}^{-1}\mathrm{Pr}^{-1}\nabla^2 T^{(v)}, \end{aligned} \tag{33}$$

where Re is the Reynolds number and Pr the Prandtl number. In the inner coordinates, these become

$$\begin{aligned} J_{CL}\left(\Psi^{(v)}, \mathcal{M}\Psi_{YY}^{(v)}\right) - \varepsilon \mathrm{J}_1 \Theta_\zeta^{(v)} &= -\lambda\mathcal{M}^2\Psi^{(v)}, \\ J_{CL}\left(\Psi^{(v)}, \Theta^{(v)}\right) &= -\lambda\mathrm{Pr}^{-1}\mathcal{M}\Theta^{(v)}, \end{aligned} \tag{34}$$

where '(v)' denotes the viscous solution, and $\lambda = \left(\varepsilon^{3/2}\alpha\mathrm{Re}\right)^{-1}$ is related to the Benney-Bergeron parameter [9], measuring the relative important of the viscous and nonlinear terms inside the critical layer. This definition of λ means that we have assumed that the Reynolds number scales like $\mathrm{Re} = \mathcal{O}\left(\varepsilon^{-3/2}\right) \gg 1$. We have also introduced the rescaled two-dimensional Laplacian, $\mathcal{M} = \partial_{YY}^2 + \varepsilon\alpha^2\partial_{\zeta\zeta}^2$. In what follows, we assume that $\lambda \ll 1$, so that viscous effects remain small even in the inner region, and expand the inner stream function and temperature as a series in λ. This expansion will be of the form

$$\Psi^{(v)} \sim \Psi + \lambda\tilde{\Psi} + \mathcal{O}(\lambda^2), \tag{35}$$

where the first term on the right-hand side of the expansion is the inviscid solution, with a similar expansion for $\Theta^{(v)}$. If we substitute the expansion in λ (35) in the viscous equation (34), at $\mathcal{O}(\lambda^0)$ we recover the inviscid case, while at $\mathcal{O}(\lambda)$ we find

$$\begin{aligned} J_{CL}\left(\tilde{\Psi}, \mathcal{M}\Psi\right) + J_{CL}\left(\Psi, \mathcal{M}\tilde{\Psi}\right) - \varepsilon\mathrm{J}_1\tilde{\Theta}_\zeta &= -\mathcal{M}^2\Psi, \\ J_{CL}\left(\tilde{\Psi}, \Theta\right) + J_{CL}\left(\Psi, \tilde{\Theta}\right) &= -\mathrm{Pr}^{-1}\mathcal{M}\Theta_{YY}. \end{aligned} \tag{36}$$

If we take the limit of the equations for the viscous correction (36) as $\varepsilon \to 0$, so that $\Psi \to \Psi_0$, which was given in (27) and we introduce $\tilde{\Psi}$ as the limit of $\tilde{\Psi}$, with similar limits for the temperature, we will find that we can recover the function $\mathcal{F}_0$ in (32) from the viscous correction to Θ_0.

If we apply this limit to the first equation in (36, with Ψ_0 given by (27), we find at leading order that

$$J_{CL}\left(\Psi_0, \tilde{\Psi}_{0YY}\right) = 0, \tag{37}$$

which has a solution

$$\tilde{\Psi}_{0YY} = \tilde{\mathcal{G}}_0(\Psi_0); \tag{38}$$

since Ψ_0 already satisfies the viscous equation (as $\Psi_{0YYYY} = 0$), we may take the function $\mathcal{G}_0$ to be zero, and consequently set $\tilde{\Psi}_0 = 0$. This result could be obtained more rigorously by also reintroducing viscosity in the outer expansion, and matching the two expansions together. If we apply the same limit to the temperature equation in (36), we find that

$$J_{CL}\left(\Psi_0, \tilde{\Theta}_0\right) = -\mathrm{Pr}^{-1}\Theta_{0YY}. \tag{39}$$

Using $\eta = \zeta$ and Ψ_0 as von Mises coordinates, we can write (39) in the form

$$\tilde{\Theta}_{0\eta} = \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial \Psi_0}\left(\mathcal{F}_0'\Psi_{0Y}\right), \tag{40}$$

which can be integrated with respect to η to give

$$\tilde{\Theta}_0 = \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial \Psi_0}\left(\mathcal{F}_0'\int_0^{\eta}\Psi_{0Y}\mathrm{d}\eta_1\right) + \mathcal{H}(\eta). \tag{41}$$

If we integrate (40 between 0 and 2π and use periodicity, we get

$$\mathcal{F}_0'\int_0^{2\pi}\Psi_{0Y}\mathrm{d}\eta = A\Psi_0 + B. \tag{42}$$

This equation holds for all values of Ψ_0, and in particular, if we evaluate it asymptotically for large values of Ψ_0, recalling that $\Psi_0 = \frac{1}{2}Y^2 + \cos\zeta - 1$, we can show that $A = 0$ and $B = 2\pi$ (whereas for the Garcia model [31], we had $A = 12\pi$ and $B = 21\pi/2$), so that

$$\begin{aligned}
\mathcal{F}_0' &= \frac{2\pi}{\int_0^{2\pi}\Psi_{0Y}\mathrm{d}\eta} = \pm\frac{\sqrt{2}\pi}{\int_0^{2\pi}\sqrt{\Psi_0 + 1 - \cos\eta}\,\mathrm{d}\eta} = \pm\frac{\pi}{2\sqrt{2}\sqrt{\Psi_0+2}\,E\left(\sqrt{2/(\Psi_0+2)}\right)},\\
\mathcal{F}_0 &= \int_0^{\Psi_0}\mathcal{F}_0'(\Psi_D)\mathrm{d}\Psi_D = \mp\pi\int_1^{\sqrt{2}/\sqrt{\Psi_0+2}}\mathrm{d}k/(k^2E(k))\\
&\sim \pm\left[(2\Psi_0)^{-1/2} - (2\Psi_0)^{-3/2} + 7\,(2\Psi_0)^{-5/2}/4 + \cdots\right],
\end{aligned} \tag{43}$$

where $E(k) = \int_0^{\pi/2}\sqrt{1 - k^2\sin^2\eta}\,\mathrm{d}\eta$ is the complete elliptic integral of the second kind. The asymptotics come from the behavior of $E(k)$ for small arguments, and matching to the outer tells us the values of some of the coefficients in the series for the mean flow distortions used in the outer expansion, $H_{10} = 0$, $H_{20} = -1/3$ and $H_{30} = 0$. It should be noticed that the temperature gradient is discontinuous at this order, since

$$\mathcal{F}_0' \to \begin{cases} \pi/4 & \text{as } \Psi_0 \to 0+ \\ 0 & \text{as } \Psi_0 \to 0- \end{cases}; \tag{44}$$

consequently, the $\mathcal{O}\left(\varepsilon^2\right)$ vorticity will also be discontinuous for $\mathrm{J}_1 \neq 0$. As Haberman [17] pointed out, a thin diffusive layer is necessary at the edge of the core to smooth out this discontinuity, and so we need to pose yet another expansion of the viscous solution, this time in ε,

$$\Psi^{(v)} \sim \Psi_0^{(v)} + \varepsilon\log\varepsilon\Psi_{1l}^{(v)} + \varepsilon\Psi_1^{(v)} + \varepsilon^2(\log\varepsilon)^2\,\Psi_{2ll}^{(v)} + \cdots, \tag{45}$$

with a similar expansion for $\Theta^{(v)}$. Once again, the log terms are necessary because they appear in the outer expansion. If we substitute this expansion in the viscous equations (34), at leading order, we get

$$J_{CL}\left(\Psi_0^{(v)}, \Psi_{0YY}^{(v)}\right) = -\lambda \Psi_{0YYYY}^{(v)}, \qquad J_{CL}\left(\Psi_0^{(v)}, \Theta_0^{(v)}\right) = -\lambda \mathrm{Pr}^{-1}\Theta_{0YY}^{(v)}. \tag{46}$$

In (46), we have taken the limit as $\varepsilon \to 0$ of the viscous inner equations (34), while in (37, 39), we applied the same limit to (36), the viscous correction to the inviscid equations. In (46), we can take $\Psi_0^{(v)} = \Psi_0$ as it satisfies the vorticity equation, just as we set $\tilde{\Psi}_0 = 0$ earlier, so that the temperature equation in (46) becomes

$$J_{CL}\left(\Psi_0, \Theta_0^{(v)}\right) = -\lambda \mathrm{Pr}^{-1}\Theta_{0YY}^{(v)}. \tag{47}$$

Once again, we will use $\eta = \zeta$ and Ψ_0 as von Mises coordinates, and to leading order we have [12]

$$\Psi_{0Y}\Theta_{0\eta}^{(v)} = \lambda \mathrm{Pr}^{-1}\left(\Psi_{0Y}^2\Theta_{0\Psi_0\Psi_0}^{(v)} + \Psi_{0YY}\Theta_{0\Psi_0}^{(v)}\right) = \lambda \mathrm{Pr}^{-1}\left(\Psi_{0Y}^2\Xi_{\Psi_0} + \Xi\right) \tag{48}$$

where we have defined $\Xi = \Theta_{0\Psi_0}^{(v)}$. Since we want the convective and diffusive terms to balance inside the diffusive layer, we shall introduce the rescaled stream function $\Upsilon = \Psi_0\sqrt{\mathrm{Pr}/\lambda}$, and then

$$\Theta_{0\eta}^{(v)} = \sqrt{\lambda \mathrm{Pr}^{-1}}\left(\Psi_{0Y}^2\Xi_{\Upsilon} + \mathcal{O}\left(\sqrt{\lambda \mathrm{Pr}^{-1}}\right)\right) \tag{49}$$

and differentiating this with respect to Υ we have

$$\begin{aligned} \Xi_\eta &= \Psi_{0Y}\Xi_{\Upsilon\Upsilon} + \mathcal{O}\left(\sqrt{\lambda \mathrm{Pr}^{-1}}\right) = \Xi_{\Upsilon\Upsilon} \pm 2\sqrt{\sin^2\eta/2}, \\ \Xi &= \frac{\pi}{4}\mathrm{erf}\left[\Upsilon\left(32\sin^2\frac{\eta}{4}\right)^{-1/2}\right] \end{aligned} \tag{50}$$

for $Y \geq 0$, and the same expression with ‘sin’ replaced by ‘cos’ for $Y \leq 0$. Having found Ξ, we can recover $\Theta_0^{(v)}$ using (49), meaning that it is possible to smooth out the discontinuity in the $\mathcal{O}(\varepsilon^0)$ temperature gradient.

Having completed our analysis of the Θ_0, we now return to our inviscid expansion (23) and consider the $\mathcal{O}(\varepsilon)$ vorticity equation,

$$J_{CL}(\Psi_0, \Psi_{1YY}) = Y\sin\zeta + \mathrm{J}_1\Theta_{0\zeta}. \tag{51}$$

At this point, we require separate solutions inside the core, where $\Theta_0 = 0$, and outside the core, where Θ_0 is given by (32, 43). Once again, we will take the solution inside the core to be the corresponding term from the Stuart vortex (1) written in the inner coordinates,

$$\Psi_{1s} = \frac{1}{4} - \frac{\cos 2\zeta}{4} + \frac{Y^2}{2}(\alpha_1 - \cos\zeta) - \frac{Y^4}{12}. \tag{52}$$

Outside the core, the stream function Ψ_{1o} obeys

$$J_{CL}(\Psi_0, \Psi_{1oYY} - \Psi_{1sYY}) = \mathrm{J}_1\Theta_{0\zeta} = \mathrm{J}_1\Psi_{0\zeta}\mathcal{F}_0'(\Psi_0), \tag{53}$$

which has a solution

$$\begin{aligned} \Psi_{1oYY} &= \Psi_{1sYY} + \mathrm{J}_1 Y\mathcal{F}_0'(\Psi_0) + \mathcal{G}_1(\Psi_0), \\ \Psi_{1o} &= \Psi_{1s} + \mathrm{J}_1\int_{\sqrt{2(1-\cos\zeta)}}^{Y}\mathcal{F}_0\left(\frac{Y_a^2}{2} + \cos\zeta - 1\right)\mathrm{d}Y_a \\ &\quad + \int_{\sqrt{2(1-\cos\zeta)}}^{Y}\int_{\sqrt{2(1-\cos\zeta)}}^{Y_b}\mathcal{G}_1\left(\frac{Y_b^2}{2} + \cos\zeta - 1\right)\mathrm{d}Y_a\mathrm{d}Y_b. \end{aligned} \tag{54}$$

In (54), $\mathcal{G}_1$ is an unknown function, which can be determined in the same manner as $\mathcal{F}_0$, by reintroducing viscosity and finding the viscous correction to Ψ_1. The reader is referred to [35] for the details of the analysis, which are somewhat involved, while the function itself is given by

$$\mathcal{G}_1 = \frac{\mathrm{J}_1 \pi}{16}\left(1 - \frac{1}{\mathrm{Pr}}\right)\left[1 - \frac{2(\Psi_0 + 1)}{(\Psi_0 + 2)E^2\left(\sqrt{2/(\Psi_0 + 2)}\right)}\right]. \tag{55}$$

Both the stream function and the velocity at $\mathcal{O}(\varepsilon)$ will be continuous across the edge of the core, but the vorticity will be discontinuous; this discontinuity arises because of discontinuity in the $\mathcal{O}\left(\varepsilon^0\right)$ temperature gradient, which is coupled back to the vorticity by the $\mathcal{F}_0'$ term in the expression for Ψ_{1oYY} in (54). Since the discontinuity in this temperature gradient will be smoothed out by the diffusive layer we introduced earlier, it follows that so will the discontinuity in this vorticity term. We also note that once again, expanding this solution as a series in the limit $\Psi_0 \gg 1$ allows us to determine more of the mean flow coefficients from the outer expansion, $G_{30} = \mathrm{J}_1(3 + \frac{1}{24}\mathrm{Pr}^{-1})$ and $\psi''_{10c} = \alpha_1 + \mathrm{J}_1\left(1 + \mathrm{Pr}^{-1}\right)\left[\frac{1}{2} + \frac{1}{16}\pi^2\right]$.

Finally, in the inner expansion, we can calculate the $\mathcal{O}\left((\varepsilon \log \varepsilon)^2\right)$ stream function, for which we will have

$$J_{CL}\left(\Psi_0, \Psi_{2llYY}\right) = 0, \tag{56}$$

which has a solution

$$\Psi_{2ll} = \mathcal{G}_{2ll}\left(\Psi_0\right), \tag{57}$$

where once again $\mathcal{G}_{2ll}$ is a general function. Since our outer solution written in inner variables,

$$\Psi_{2ll} = \frac{\mathrm{J}_1^2}{8}\left(\cos\zeta - 1\right), \tag{58}$$

satisfies (57), we are able to use (58) throughout the entire inner region, including inside the core; by comparison, the Stuart vortex had no term at this order, and once again, this term vanishes when $\mathrm{J}_1 = 0$.

To close this section, in Figures 1–3, we show our solutions for the case of Prandtl number equal to unity, and also set $\varepsilon = 0{\cdot}1$, $\mathrm{J}_1 = 1$ (so that the Richardson number is 0·1, a value that is typical of simulations) and $\alpha_1 = 1$. In Figure 1, we plot the contours of constant vorticity; the edge of the core is shown as a dashed line. In this plot, we have included the vorticity terms as far as $\mathcal{O}\left((\varepsilon \log \varepsilon)^2\right)$ terms, so that we are plotting

$$-\left(\Psi_{0YY} + \varepsilon\left[\log\varepsilon\Psi_{1lYY} + \Psi_{1sYY} + \Psi_{0\zeta\zeta}\right] + \varepsilon^2\left[(\log\varepsilon)^2\,\Psi_{2llYY} + \log\varepsilon\left(\Psi_{2lYY} + \Psi_{1l\zeta\zeta}\right)\right]\right) \tag{59}$$

inside the vortex cores, while outside the cores, we show

$$-\left(\Psi_{0YY} + \varepsilon\left[\log\varepsilon\Psi_{1lYY} + \Psi_{1oYY} + \Psi_{0\zeta\zeta}\right] + \varepsilon^2\left[(\log\varepsilon)^2\,\Psi_{2llYY} + \log\varepsilon\Psi_{1l\zeta\zeta}\right]\right). \tag{60}$$

The most notable feature of this plot is the structure around the edges of the core: the cat's eyes have 'eye-lids' in the stratified case which are not present in the unstratified Stuart vortex. For comparison purposes, we also show the vorticity pattern for a solution with a vorticity-homogenized core in Figure 1: the solution is the same outside the core and differs only inside the core where (59) is replaced by a constant. The vorticity section shown in Figure 3 can be

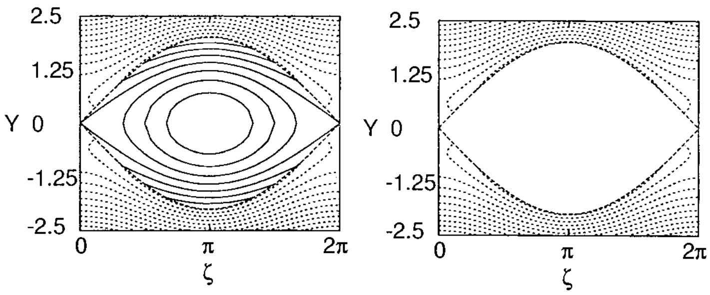

Figure 1. Contours of constant vorticity for both our solution and a vorticity-homogenized (BBD) core.

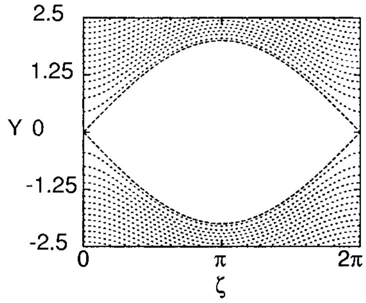

Figure 2. Contours of constant temperature for our solution.

compared to the vorticity surface shown in [2, Figure 3]. The cut at the rear of their surface corresponds to our section through the core, and it can be seen that the gross features are very similar, with our section including the round hill of vorticity in the core together with the ridge-like structure of the braids, with the two separated by a low valley. Similarly, our temperature section compares very well with the density section plotted in [3, Figure 15(b)]. For the latest time shown in their figure ($\tau = 4$), the density inside the core appears to be uniform, while there is a fairly steep density gradient at the edge of the core; both of these features are also present in our plot.

In Figure 2, we show the contours of constant temperature, with the edge of the core again shown as a dashed line. The temperature is constant inside the core, while outside the core, we have plotted Θ_0 as given by (32,43)., In Figure 3, we show the vorticity and temperature on a section through the core, on the line $\zeta = \pi$. For the vorticity section, we have included terms as far $\mathcal{O}\left((\varepsilon \log \varepsilon)^2\right)$, as for Figure 1, so that we are plotting (59) inside the core and (60) outside. In the plot of the vorticity, for comparison purposes, we show a vorticity-homogenized core as a dashed line, which is again obtained simply by replacing the vorticity inside the core (59) by a constant, while the unstratified case, which is obtained by setting $J_1 = 0$ in (59,60) is marked with diamonds. The discontinuity in the vorticity at the edge of the core (the eyelid in Figure 1) is clearly visible in this figure. Finally, in the temperature section, we again plot Θ_0 outside the core and while the temperature is set to zero inside the core.

3. Discussion

In this paper we have used nonlinear critical layer techniques to find nonlinear steady state vortex solutions for the Holmboe model of a stratified shear layer; these solutions represent the quasi-equilibrium large scale vortices that emerge following vortex roll-up in a mixing

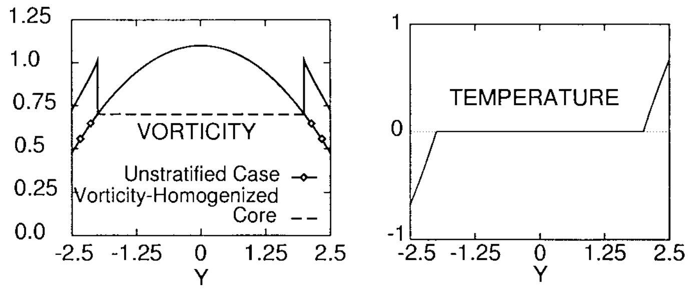

Figure 3. Vorticity and temperature on the line $\zeta = \pi$ through the core.

layer. These solutions differ from those previously presented by Maslowe [12] and Haberman [16, 17] in that the vorticity inside the core is based on the Stuart vortex [8] rather than the BBD-style vorticity-homogenized cores [9, 10] utilized by those authors, while the solution outside the core is essentially the same as that in [17]. Our reasons using the Stuart vortex as the solution inside the core were presented in the introduction and are essentially due to a combination of Brown and Stewartson's argument that the cat's eyes would decay before vorticity has had time to homogenize [15], which has been confirmed by numerical simulations, and the claim by Sommeria *et al.* [4] that the Stuart vortex is a preferred state on equilibrium grounds.

Turning to the figures in Section 2, which present our solution graphically, the striking feature of the vorticity contours (and also the vorticity section) is the presence of the 'eye-lid' which in absent in the unstratified case. The existence of this eye-lid has been confirmed in simulations [5, 6], with very sharp gradients of vorticity being found at the edge of the core, and it arises because of discontinuities in the temperature gradient at the edge of the core (44). The sections which we presented in Figure 3 compare very well with the predictions of numerical simulations: the vorticity section appears very similar to a cut through the vorticity section of [2, Figure 3], with both figures sharing the round hill of vorticity in the core together with the ridges of vorticity in the braids and a valley between the hill and the ridges, while our temperature section is very similar to the section presented in [3, Figure 15(b)], with the density inside the core being uniform in both figures.

Finally, we make a suggestion for future work. The present analysis has been carried out for the slightly stratified case, which we have shown [27] is appropriate for the initial roll-up stage, when it appears that the Miles phase shift [25] is absent. However, it is believed that there is a secondary stage [5] following the initial roll-up in which both the $\alpha_0 = 1/2 + \sqrt{1/4 - \mathrm{J}_0}$, and the $\alpha_0 = 1/2 - \sqrt{1/4 - \mathrm{J}_0}$ neutral modes are present and the Miles phase shift occurs across the critical layer, and in this secondary stage, it is not appropriate to use the slightly stratified approach and the Richardson number J_0 must be regarded as order one. It would be interesting to see if a steady state solution could be found for this secondary, strongly stratified stage with both neutral modes present.

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